Mathematics of knowledge spaces

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Overview

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- Several equivalent ways of characterizing knowledge spaces
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Definition, examples, and arguments

Given a set Q of

- test items of of an achievement test, or
- problems to be solved in a subject in school, or
- items in a questionnaire.

Any element in Q can be valuated by

- $0/1$, or
- true/false, or
- correct/incorrect, or
- agree/disagree.

Definition, examples, and arguments II

A knowledge space is a pair $(Q, \mathcal{K})$ consisting of a set Q of items, problems, tasks etc. and a family $\mathcal{K}$ of subsets of Q fulfilling

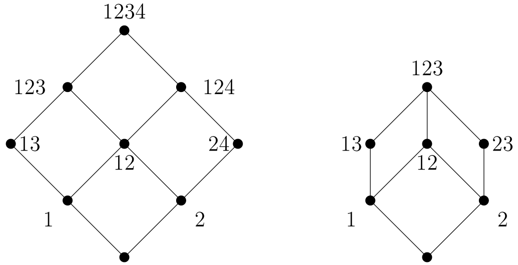
1. $\varnothing, Q \in \mathcal{K}$
2. closure under union, i.e.,

$$K, L \in \mathcal{K} \quad \text{implies} \quad K \cup L \in \mathcal{K}.$$

Any $K \in \mathcal{K}$ is a learning state.
Let's regard only finite Q

Two examples

The space on the left is on $Q = \{1, 2, 3, 4\}$ the space on the right on $Q = \{1, 2, 3\}$



More on nomenclature and conventions

The elements of $\mathcal{K}$ are called *states* or learning states, a maximal chain in $\mathcal{K}$ is called a *learning path*.

A knowledge space is called *discriminative* if the sets

$$\mathcal{K}_q := \{K;\ q \in K\}$$

are all different, i.e., $\mathcal{K}_q \neq \mathcal{K}_p$ for all $p, q \in Q$ with $p \neq q$. Clearly, it is no big deal to make a knowledge space discriminative if it is not by deleting redundant elements from Q. So we can always assume that $\mathcal{K}$ is discriminative.

Why these conditions?

Q and $\varnothing$ are possible learning states, state Q is knowing everything, state $\varnothing$ corresponds to knowing nothing at all.

Why is $K \cup L$ a state when K, L are states?
Combining the knowledge of two subjects, one in state K, the other in state L is assumed by Condition 2 to result in a legitimate new knowledge state.

Closure under intersection?

One might wonder why not closure under intersection is stipulated in addition to closure under union.
The reason is mostly empirical: Many of the situations which are intended applications — mostly learning situations — fail to satisfy it.

If a knowledge space satisfies

$$K, L \in \mathcal{K} \Rightarrow K \cap L \in \mathcal{K}$$

i.e., closure under intersection then it is called *quasi-ordinal* and *ordinal* when it is discriminative.

Well-gradedness — Learning spaces

However another condition seems rather natural from a pedagogical point of view:
A knowledge space is called *well-graded* or a *learning space* if for all $K, L \in \mathcal{K}$ with $K \subset L$ there are elements $q_1, \ldots, q_n \in Q$ such that

1. $K \cup \{q_1\},\ K \cup \{q_1, q_2\},\ \ldots\ K \cup \{q_1, \ldots, q_n\} \in \mathcal{K}$
2. $L = K \cup \{q_1, \ldots, q_n\}$

In a well-graded knowledge space the items are chosen in such a way that the learning progress is rather 'smooth'.

General opinion is that learning spaces strike the right balance between the generality of knowledge spaces and specificity of ordinal spaces. We shall come back to this issue later.

Several equivalent ways of characterizing knowledge spaces

- Characterization by bases
- Characterization by entailment
- Characterization by surmise functions
- Characterization by fringes

Characterization by the basis

A **basis** is a minimal subset of states $\mathcal{B}$ of $\mathcal{K}$ which when closed under union yields all of $\mathcal{K}$.

Each knowledge space $\mathcal{K}$ admits a unique basis.
In consequence, all the information on $\mathcal{K}$ is contained in $\mathcal{B}$.
Furthermore, $\mathcal{B}$ is usually much smaller than $\mathcal{K}$. This is a valuable property given the enormous number of states knowledge spaces nowadays in use usually have.

Moreover, the basis elements carry an **empirical meaning:**
Each $B \in \mathcal{B}$ can be identified with a skill of the body of knowledge captured by $\mathcal{K}$.
This property can be exploited.

Details on the basis

Let $K \in \mathcal{K}$ be a knowledge state in a space $\mathcal{K}$, and $q \in K$.
We call K an *atom at* q if no subset of K which contains q is a state of $\mathcal{K}$.
Note: K may contain smaller states, but they do not contain q.
The set of all states which are atoms at some element of Q can be shown to be a basis of $\mathcal{K}$.
Once the fact that this set is a basis is established the uniqueness is straightforward because the atoms at some q are necessarily base elements.

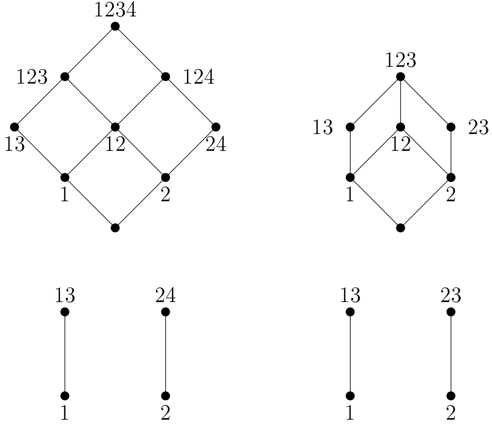
Comparison to other basis concepts in mathematics

You may be familiar with the basis of a vector space. Every vector can be represented as a linear combination of basis vectors and no subset of the basis has this property.

Both conditions are shared by the basis of a knowledge space if we replace "linear combination" by "set theoretic union". However, a vector space has many different bases, while a knowledge space has only one basis.
In both cases the basis concepts are of great importance.

The bases of the two examples

The two knowledge spaces on page 5 have the following bases:



Advantages of dealing with the basis

From Linear Algebra or Functional Analysis we know how important and useful working with a basis is. This is also true of the basis of a knowledge space — albeit for partly different reasons. Note that all the information on $\mathcal{K}$ is contained in the basis.

Reason 1 for utilizing the basis

The number of states of knowledge spaces dealt with in practice is usually very large. Today, situations with $|Q|$ between 20 and 300 are quite normal.
Note that the largest knowledge space is the powerset of Q. Thus,

$$|\mathcal{K}| \leq 2^{|Q|}.$$

In interesting applications $|\mathcal{K}|$ is smaller, however still very large — too large to be handled explicitly. The number of states reaches from a few thousands to tenth of millions.

The number of basis elements is usually much smaller than $|\mathcal{K}|$.

Related to this observation is an **open problem:** How large can a basis maximally be in terms of $|Q|$?

Reason 2 for utilizing the basis

The basis elements have an empirical meaning which can be formulated in terms of the area of knowledge described by $\mathcal{K}$. They can be associated with *skills*.

We will deal with this interpretation later.

Exercise

Construct all knowledge spaces on a set Q of cardinality 2 and 3.

Note: It is always convenient to assume that the knowledge space is discriminative, i.e., no two elements in Q are always in the same states. However, for this exercise count also the non-discriminative spaces. Furthermore, determine how many are well-graded, quasi ordinal, ordinal.

Characterization by entailment

Given a set of items Q. A relation $\mathcal{E}$ on $2^Q - \{\varnothing\}$ is called an *entail relation* for Q when it satisfies

1. $A \supseteq B$ implies $A \, \mathcal{E} B$;
2. $\mathcal{E}$ is transitive, i.e., $A \, \mathcal{E} B$ and $B \, \mathcal{E} C$ implies $A \, \mathcal{E} C$;
3. If $A \, \mathcal{E} B_i$ for $i = 1, \ldots, n$ then $A \, \mathcal{E} \left(\bigcup_{i=1}^n B_i\right)$

Entail relations and knowledge spaces are related via the following two equivalences:

$$A \, \mathcal{E} q \quad \Leftrightarrow \quad (\forall K \in \mathcal{K} : \; A \cap K = \varnothing \Rightarrow q \notin K)$$
$$K \in \mathcal{K} \quad \Leftrightarrow \quad (\forall (A, p) \in \mathcal{E} : \; A \cap K = \varnothing \Rightarrow p \notin K)$$

Entailment in practice

In practice entail relations enter the picture by eliciting answers from an expert of a body of information for which a knowledge space construction is under way to questions like this one

Suppose that a student has failed to solve items $p_1, \ldots, p_n$. *Do you believe that this student would also fail to solve item* q.

Such a question is part of a query procedure. Asking all such questions is clearly too time consuming.
Koppen constructed a procedure (called QUERY) in which cleverly selecting questions and inferring the answers to the other questions is implemented. This is a powerful tool to knowledge space construction for moderately large item sets Q.

Entailment with $n = 1$

Asking only the questions with $n = 1$, i.e., asking

Does failing p imply failing q?

is not sufficient to generate a general knowledge space. It were sufficient if we knew in advance that the space is ordinal. In general such an assumption is hard to defend.
However, it would simplify matters.

Characterization by a surmise function

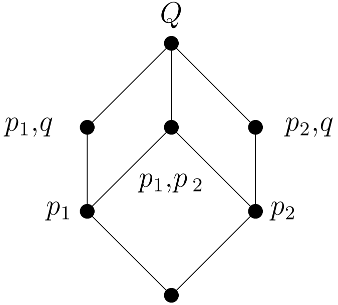
A function $\sigma : \; Q \rightarrow 2^{2^Q}$ is a *surmise function* if the following four conditions are satisfied

1. $\sigma(q) \neq \varnothing$;
2. if $C \in \sigma(q)$ then $q \in C$;
3. if $p \in C \in \sigma(q)$ then $D \subseteq C$ for some $D \in \sigma(p)$;
4. if $C, D \in \sigma(q)$ and $D \subseteq C$ then $D = C$.

Interpretation:
Each $C \in \sigma(q)$ is a set of items which constitute a learning history; knowledge of q can go along with any of the sets in $\sigma(q)$. In the example corresponding to the figure on the next page we have $\sigma(q) = \{\{p_1, q\}, \{p_2, q\}\}$. Needless to say, however, the student can first learn p_1 and p_2 before embarking on q.

Example of a surmise function

$$\begin{aligned}\sigma(p_1) &= \{\{p_1\}\}\\ \sigma(p_2) &= \{\{p_2\}\}\\ \sigma(q) &= \{\{p_1, q\}, \{p_2, q\}\}\end{aligned}$$



Note on prerequisite and surmise relation

If we just ask whether an item q is a prerequisite for item p then we have a ‘little sister’ of a surmise function, called a *surmise relation*.
It is a partial order.

However, the surmise relation is not sufficient to construct the knowledge space.
If we do it based only on this information the ensuing space is necessarily ordinal.

Surmise relation of the example

In the example of the previous Figure we easily see that none of the three elements has another item as a prerequisite. Thus, the surmise relation is such that all three elements are incomparable, or formulated as a surmise function we have:

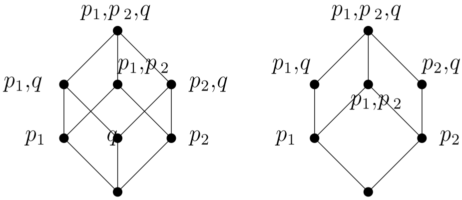
$$\sigma(p_1) = \{p_1\},\ \sigma(p_2) = \{p_2\},\ \sigma(q) = \{q\}$$

The corresponding knowledge space is the power set of $\{p_1, p_2, q\}$.

Ordinal spaces tend to be too large

The fact that the space derived from the simpler surmise relation is larger than the one derived from the full fledged surmise function is characteristic of this situation.

Usually, the ordinal space based on the surmise relation is much larger than the one based on a more sophisticated surmise function.



Characterization by the fringes

Given a state $K \in \mathcal{K}$. The *outer* fringe of K is the set of all items $q \in Q$ which are not in K but in some upper neighbor of K.
The outer fringe of K is denoted by $K^{\mathfrak{O}}$

The *inner* fringe of K is the set of items $q \in K$ which are not in any state $L \subset K$.
The inner fringe is denoted by $K^{\mathfrak{I}}$.

The union of $K^{\mathfrak{O}}$ and $K^{\mathfrak{I}}$ is the *fringe* of K denoted by $K^{\mathfrak{F}}$.

The meaning of the fringes is important in practice. It describes what a student in state K has just learned and should learn next.

Results on fringes

Learning spaces enjoy an important property with respect to their fringes:

Theorem

Let $\mathcal{K}$ be a knowledge space. The following three conditions are equivalent.

- $\mathcal{K}$ is a learning space
- for all $K, L \in \mathcal{K}$

$$((K - L) \cup (L - K)) \cap K^{\mathfrak{F}} \neq \varnothing$$

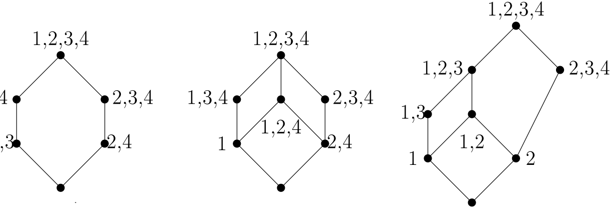
- for all $K, L \in \mathcal{K}$ we have

$$K^{\mathfrak{I}} \subseteq L \quad \text{and} \quad K^{\mathfrak{O}} \subseteq Q - L \quad \text{implies} \quad K = L$$

In particular, the third expression is useful in practice. It says that in a learning space a state is fully specified by its two fringes.

Example. Violation of well-gradedness read off from the fringe

Here are three examples of not well-graded spaces



a **b** **c**

Violation of the conditions in the examples

	$L \cap K^{\mathfrak{I}} \subseteq L,\ K^{\mathfrak{O}} \subseteq \overline{L},\ K \neq L$	$((K - L) \cup (L - K)) \cap K^{\mathfrak{F}} = \varnothing$
a	$K = \{1,3\},\ L = \{1,3,4\}$	$K = \{2,4\},\ L = \varnothing$
b	$K = \{2,4\},\ L = \{1,2,4\}$	$K = \{2,4\},\ L = \varnothing$
c	$K = \{2\},\ L = \{2,3,4\}$	$K = \{2,3,4\},\ L = \{2\}$

Open problems with fringes

Given a collection, say $\mathcal{F}$, of subsets of Q. What are the conditions on this family to make them the fringes of a knowledge space?

If $\mathcal{F} = \mathcal{I} \cup \mathcal{O}$, what are the conditions on $\mathcal{I}$ and $\mathcal{O}$ such that they are the inner and outer fringes of a knowledge space.

What is the analogue of Theorem 28 for general knowledge spaces?

Knowledge spaces as lattices

KST can be formulated with lattice theoretic terminology. We briefly introduce this important concept.

Definition

Let L be a set and $\preccurlyeq$ a binary relation on L. The pair $(L, \preccurlyeq)$ is a *partial order* if it satisfies

- reflexivity, i.e., $a \preccurlyeq a$,
- antisymmetry, i.e., $a \preccurlyeq b$ and $b \preccurlyeq a$ implies $a = b$,
- transitivity, i.e., $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $a \preccurlyeq c$.

A partial order is a lattice if for all $a, b \in L$ a least upper bound, i.e., a supremum, denoted by $a \vee b$ and a greatest lower bound, i.e., an infimum, denoted by $a \wedge b$ exists.

Importance of lattice theory

Lattices and lattice theory form an important link in modern mathematics between various branches of mathematics and science, such as topology, algebra, functional analysis, quantum mechanics, discrete mathematics, order theory.

'Classical' books on lattice theory

Birkhoff, G. (1967) *Lattice Theory*. Third edition. Amer. Math. Soc., Providence R.I.

Grätzer, G. (1998) *General Lattice Theory*. Second edition. Birkhäuser Verlag Basel.

Davey, B. A., & Priestley, H. A. (1990) *Introduction to Lattices and Order.* Cambridge University Press, Cambridge.

Knowledge spaces as lattices (continued)

For each finite knowledge space $\mathcal{K}$ the pair $(\mathcal{K}, \subseteq)$ is a lattice. Clearly, it is a partial order. Furthermore, we define

$$K \vee L := K \cup L$$

and

$$K \wedge L := \bigvee_{M \in \mathcal{K}, M \subseteq K, L} M.$$

For finite $\mathcal{K}$ the supremum in the last definition always exists.

Now it is routine to show that $(\mathcal{K}, \subseteq)$ endowed with supremum and infimum as defined above is in fact a lattice.

Birkhoff's Theorem

Birkhoff, one of the founders of lattice theory, proved a famous theorem, published 1937.

Theorem

There exists an bijective correspondence between the collection of all ordinal spaces $\mathcal{K}$ on Q and the collection $\mathcal{P}$ of all partial orders on Q. This correspondence is given by

$$
\begin{aligned}
p\mathcal{P}q &\Leftrightarrow \forall K \in \mathcal{K}: \; q \in K \Rightarrow p \in K \\
K \in \mathcal{K} &\Leftrightarrow \forall (p,q) \in \mathcal{P}: \; q \in K \Rightarrow p \in K
\end{aligned}
$$

If we replace partial order by quasi order then the same equations yield a correspondence between quasi orders and quasi ordinal spaces.

Generalization of Birkhoff's Theorem

Doignon and Falmagne (1985) derived the following generalization of Birkhoff's theorem.

Theorem

There is an bijective correspondence between the collection of all knowledge spaces. It is defined for all knowledge spaces $\mathcal{K}$ and all surmise functions σ by the equivalence:

$$S \text{ is an atom at } q \Leftrightarrow S \in \sigma(q)$$

where $S \subseteq Q$ and $q \in Q$.

Skills and competencies

Now we want to add a bit psychology to the picture.

What are the competencies which enable a person to solve the problems of Q?

Can they be traced in a knowledge space?

If so, how?

Competence based knowledge space theory

In

Heller, J., Steiner, C., Hockemeyer, C. & Albert, D. (2006). Competence-based knowledge structures for personalised learning. International Journal on E-Learning, 5, 75-88.

we find as a summary of what can be called the approach of the **Graz school**. It is an extension of knowledge space theory, mostly motivated to adapt it to personalized learning, preferably in an automated setting.

Ingredients of CbKST

Three different entities are introduced:

1. a set Q of assessment problems,
2. a set L of learning objects, and
3. a set S of competencies which appear relevant to solve the problems in Q and which are taught by the learning objects.

Q gives rise to a knowledge space in the usual sense.

The structure on S is a knowledge space formed with the competencies in S.

The elements of S and the structure on S are derived in a way which is in some cases fundamentally different from that on Q. A so called 'concept map' is derived.

All the notions of a field of knowledge and their relationships are gathered in it.

Next a hierarchical structure is derived from this information.

The competence structure is kind of a knowledge space which respects this information.

Remarks on CbKST

A space of competencies and knowledge spaces live in parallel universes.
They are linked by so called skill maps.

$$(C, \mathcal{C}) \quad (Q, \mathcal{K})$$

Although there are many questions left unanswered within CbKST and its relation to 'classical' knowledge space theory, it has no doubt triggered many successful applications.

The main advantage seems to be its more direct link to pedagogical and psychological theories.

To witness see the numerous applications of the Graz group.

Skills in classical Knowledge Space Theory

Skills enter the picture of KST via *skill maps*, cf. Falmagne & Doignon (2011), ch. 4.
A skill map is a triple

$$(Q, S, \tau)$$

where Q is an item set, S is a skill set, and τ a map such that

$$\tau : \; Q \longrightarrow 2^S - \{\varnothing\},$$

with the understanding that $\tau(q)$ is the set of skills assigned to q.
Such a structure *delineates* a knowledge space in the sense that any $T \subseteq S$ defines a knowledge state K by

$$K = \{q \in Q;\; \tau(q) \cap T \neq \varnothing\}. \tag{1}$$

Skills in classical Knowledge Space Theory II

A skill map is called *minimal* if the omission of any element in S delineates a non isomorphic knowledge space according to (1).

In Chapter 4 of Falmagne & Doignon (2011) the following important result is proven:
A minimal skill map corresponds to the basis, i.e., the sets

$$K(s) := \{q \in Q;\ s \in \tau(q)\}$$

form the basis of the knowledge space $\mathcal{K}$ delineated by (Q, S, τ).

The main theorem on skill maps

This Theorem reads as follows

Theorem

A knowledge space is delineated by some minimal skill map if and only if it admits a base. The cardinality of the base equals the set of skills.

Moreover, any two minimal skill maps delineating the same knowledge space are isomorphic. Also any skill map (Q, S, τ) delineating a space $(Q, \mathcal{K})$ having a base prolongs a minimal skill map.

In particular, this theorem teaches us two lessons:

- Knowing the skills and the skill maps amounts to knowing the space.
- The base of a knowledge space has an empirical meaning.

Set representations of partial orders

Let $\mathcal{B}$ be the basis of a knowledge space. With set inclusion it is a partial order.

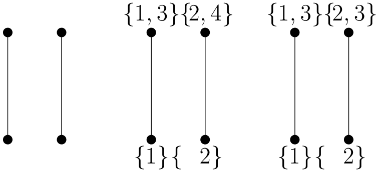
$$(\mathcal{B}, \subseteq).$$

I distinguish two aspects:

1. The abstract partial order $(P_{\mathcal{B}}, \leq)$, i.e., the order type of $(\mathcal{B}, \subseteq)$.
2. Its representation by subsets of the item set Q.

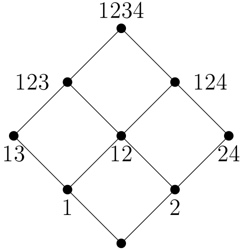
Example

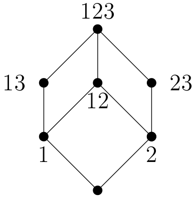
Here is a partial order **2+2** and two set representations. (The curly brackets are added this time to emphasize that we are dealing with sets).

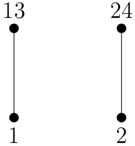


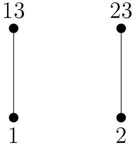
The knowledge spaces of the example

We saw the corresponding knowledge spaces already several times. (The curly brackets are omitted as in earlier figures)









The formal definition

We assume a partial order $(S, \leq)$ and a function mapping S into the power set of Q, i.e.,

$$\varphi : \; S \to 2^Q.$$

φ is called a *set representation* of $(S, \leq)$ if it satisfies:

$$s_1 \leq s_2 \quad \text{iff} \quad \varphi(s_1) \subseteq \varphi(s_2). \tag{2}$$

Thus, the partial order $(S, \leq)$ is 'represented' as a suborder of the lattice of subsets of Q.

Theorem

These set representations of partial orders are closely related to knowledge spaces. We prove

Theorem

Given a partial order $(S, \leq)$, a finite set Q and a function $\varphi : S \to 2^Q$ satisfying (2). Then

$$\left(\bigcup_{s \in S} \varphi(s), \;\; \{\varphi(s); s \in S\}^* \right)$$

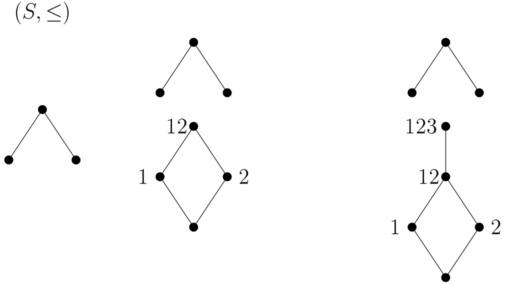
is a knowledge space with basis $\{\varphi(s); s \in S\}$ if and only if φ satisfies

$$|\varphi(s)| > \left| \bigcup_{t < s} \varphi(t) \right| . \tag{3}$$

Functions φ which fulfill (3) are called 'basic set representations' because of this theorem.

Attention please

The main point of the preceding theorem is not that $(\bigcup_{s\in S}\varphi(s), \{\varphi(s); s \in S\}^*)$ is a knowledge space — this is trivial — but rather that its basis is $\{\varphi(s); s \in S\}$.



To summarize

So far we have the two facts:

- The sets $\varphi(s)$ form the basis of a knowledge space
- The basis corresponds to the skills

These facts give rise to a somewhat different approach to knowledge space theory:

A different point of view — a new way into knowledge spaces

Because of the preceding Theorem we can think of knowledge spaces as a triple

$$((S, \leq), Q, \varphi)$$

where $(S, \leq)$ is a partial order, Q a finite set and φ a basic set representation of $(S, \leq)$, such that

$$\bigcup_{s \in S} \varphi(s) = Q$$

i.e., a mapping $\varphi : S \to 2^Q$ satisfying (2) and (3).

The set $\mathcal{K} = \{\varphi(s);\ s \in S\}^*$ is the set of knowledge states and $(Q, \mathcal{K})$ is the space, its basis is $\{\varphi(s);\ s \in S\}$.

The set S is interpreted as the skills which are pertinent to the field of knowledge which is described by the space.

Small is beautiful — large is necessary. Meshing knowledge spaces

How to construct large spaces from small ones?

This question is relevant as

- a body of knowledge may be naturally subdivided into smaller subareas for which knowledge spaces can be constructed separately;
- the set of items might be too large and the set of states even much larger such that the construction, for example by querying experts, takes too long or is so unreliable, thus rendering it unfeasible;
- in the praxis of personality tests the technique of deriving subscales is common and their interpretation furnishes a substantial part of this theory.

Thus a combinatorial analogue is called for.

Meshing spaces

A knowledge space $(Q', \mathcal{K}')$ is the projection of a space $(Q, \mathcal{K})$ if $Q' \subseteq Q$ and $\mathcal{K}'$ consists of all sets $K' = K \cap Q'$ for all $K \in \mathcal{K}$.
A knowledge space $(Q, \mathcal{K})$ is called a mesh of spaces $(R, \mathcal{R})$ and $(S, \mathcal{S})$

- $Q = R \cup S$
- $\mathcal{R}$ and $\mathcal{S}$ are projections of $\mathcal{K}$ on R and S, respectively

Not any two spaces are meshable. There must be some kind of compatibility fulfilled. The following theorem clarifies this issue

Theorem on meshability of spaces

Theorem Two knowledge spaces $(R, \mathcal{R})$ and $(S, \mathcal{S})$ are meshable if and only if for any $X \in \mathcal{R}$ the intersection $X \cap S$ is of the form $Y \cap R$ for some state $Y \in \mathcal{S}$, (the trace of some state of $\mathcal{S}$ on S).

The maximal mesh

Let $(R, \mathcal{R})$ and $(S, \mathcal{S})$ be two meshable knowledge spaces. The maximal mesh of $\mathcal{R}$ and $\mathcal{S}$ is defined to be

$$\mathcal{R} \star \mathcal{S} := \{K \in 2^{R \cup S};\ K \cap R \in \mathcal{R},\ K \cap S \in \mathcal{S}\} \qquad (4)$$

Equivalently

$$\mathcal{R} \star \mathcal{S} = \{X \cup Y;\ X \in \mathcal{R},\ Y \in \mathcal{S} \text{and} X \cap S = Y \cap R\} \qquad (5)$$

Under suitable meshability constraints one has an associative law for maximal meshes:

$$(\mathcal{K} \star \mathcal{L}) \star \mathcal{M} = \mathcal{K} \star (\mathcal{L} \star \{M\})$$

Outlook

The previous theorem is nice, but we are haunted by questions of meshability.

For this very reason we are developing a theory — based on set representations — in which meshability is replaced by order theoretic addition and product formation of the bases of smaller spaces to build up larger ones.
This is work in progress.

A few remarks on learning spaces and ordinal spaces

We already mentioned that knowledge spaces or sometimes too general. On the other hand ordinal spaces are in most cases too restricted and suffer from the property that they tend to overestimate the number of states necessary too describe a body of knowledge.
Learning spaces (well-graded knowledge spaces) seem to be just what is needed in most cases. Here are characterizations of learning spaces and ordinal spaces. (cf. Suck (2003, 2004, 2011)).

Several characterizations

- Let $\mathcal{K}$ be a discriminative knowledge space with basis $\mathcal{B}$. Then $\mathcal{K}$ is well-graded if and only if $(\mathcal{B}, \subseteq)$ is a *parsimonious* set representation of its abstract partial order $(P_{\mathcal{B}}, \leq)$. Parsimony is the property that a basic set representation uses the minimal number of elements, i.e., φ is a *parsimonious* set representation if for all $p \in P$:

$$|\varphi(p)| = |\bigcup_{q<p} \varphi(q)| + 1. \tag{6}$$

- $\mathcal{K}$ is an ordinal knowledge space iff $(\mathcal{B}, \subseteq)$ is the principal ideal set representation of $(P_{\mathcal{B}}, \leq)$, i.e.,

$$(\mathcal{B}, \subseteq) = (\nabla_{(P_{\mathcal{B}}, \leq)}, \subseteq).$$

- Let $\mathcal{K}$ be a well-graded knowledge space. If $(P_{\mathcal{B}}, \leq)$ is saturated, then $\mathcal{K}$ is ordinal.

Koppen's characterization of learning spaces

Koppen (1998) proved that the surmise function of a learning space has specific property

Theorem
A knowledge space $\mathcal{K}$ is a learning space if and only if its surmise function σ satisfies

$$\sigma(p) \cap \sigma(q) = \varnothing$$

for all $p, q \in Q$ with $p \neq q$.

In other words, any atom of $\mathcal{K}$ is an atom at only one $q \in Q$.

These results are useful in the difficult problem of constructing a learning space.

Final remarks

There are many topics which we did not mention here, such as

- Probabilistic knowledge spaces
- Stochastic learning paths
- Techniques of building a knowledge space or a learning space for large item sets
- Assessing the knowledge state of a student
- Connection to other theories such as media theory or concept analysis.

These topics and many more are currently investigated in the scientific community. It is an active field of research.

Thank you for your attention